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Beyond Flatland: The Future of Space and Time

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Excitement has been generated by the idea that the puniest of all forces, gravity, may in fact be as strong as nature's other three fundamental forces: the strong force which binds protons and neutrons together in atomic nuclei, the weak force which governs radioactive decay, and the forces that govern electricity and magnetism. The perceived mismatch between these three forces and gravity creates a theoretical nightmare; it is the principal reason we have yet to find a grand unified theory. However, it has recently been hypothesized that this weakness is a mirage: the force of gravity only appears weak because its force is diluted in our own universe and most of gravity's force radiates out into extra dimensions. All other forces remain trapped in our three-dimensional world, while gravity is free to roam other dimensions. With this hypothesis, there could be other worlds that are parallel to our own; they all neatly stack up, each oblivious of the other, with gravity the only force that moves between them. This would also account for the missing dark matter of our universe; it actually resides in other parallel universes.

New mathematics will be generated to further explore these ideas; well-developed mathematics will also be called into play. Even before this new idea, much of the relevant mathematics stirred enormous interest in the mathematical community; it already has exposed the special nature of dimensions three and four.

The basic subjects for this study are objects called manifolds. Manifolds are spaces which at any point look like Euclidean space. The dimension of a manifold is the dimension of this Euclidean space. While locally manifolds are very familiar, their global structure can be unimaginably complex. Manifolds permeate much of mathematics, physics, and engineering. Manifolds appear as the space of solutions to differential equations, as the phase space for physical problems and phenomena, and as the domain in which one studies the theory of differential equations. We live in a world that at any point looks like three-dimensional Euclidean space; with time included, we live in a world that at any point looks like four-dimensional Euclidean space.

A natural task is to classify n-dimensional manifolds; i.e., for a given integer n, give a list of all n-dimensional manifolds. The circle is an example of a one-dimensional manifold and is, in fact, the only one-dimensional manifold that is compact, i.e., can be covered by a finite number of Euclidean neighborhoods. Familiar two-dimensional compact manifolds are the sphere (e.g., the surface of the earth), the torus (e.g. a coffee-cup with one handle), or a sphere with several handles (e.g., coffee-cup with many handles). Other less obvious examples of two-dimensional compact manifolds are the collection of lines in Euclidean three-space which go through the origin (called the projective plane), and the projective plane with handles attached to it like handles can be attached to a coffee-cup. These, in fact, are the only two-dimensional compact manifolds. While these low dimensional manifolds are still easily imagined, the situation is much more complex in higher dimensions.

Exciting and deep mathematical work during 1960-1975 accomplished the remarkable feat of classifying manifolds in every dimension greater than four. However, this feat turns into defeat in dimensions three and four; the techniques used do not work in these physically important dimensions. Roughly speaking, one only needs to understand the properties of a manifold up to half its dimension; a duality theory shows that for an n-dimensional manifold, properties that can be verified in r dimensions are also true in (n-r) dimensions. There is one catch--a knotting phenomenon occurs in dimension (n-2) analogous to the knotting of circles in the three-dimensional sphere. Thus, in order to understand four-dimensional manifolds we need to understand phenomena up through dimension 4/2=2, but this is precisely the dimension in which knotting can occur (2=4-2). Similarly, for three-dimensional manifolds, knotting obstructs understanding their one-dimensional properties.

It has been the very exciting and much deeper mathematics of the last twenty years that has attempted to understand these three- and four-dimensional manifolds. However, the techniques used for these two dimensions have diverged. In dimension three there is a conjectured classification: every three manifold can be cut along two-dimensional tori so that the resulting pieces possess one of the eight classical geometries in dimension three. This geometrization conjecture is the target of many international research groups.

The situation is far worse in dimension four; there is not even a conjectured classification of four-dimensional manifolds. An arsenal of techniques has been thrown at this problem; it is the focus of dozens of international research groups. The most successful attempts have associated to each four-dimensional manifold the solution space to complex systems of equations that arise in particle physics: the Yang-Mills equations and the monopole equations of Seiberg and Witten. These solution spaces are useful in distinguishing cunningly constructed four-dimensional manifolds. The result of this assault is that four-dimensional manifolds are more complicated than we ever expected. The discovery of large new families of four-dimensional manifolds has led many researchers to believe that we have yet to find the mother lode. As a result, it is impossible to predict a classification scheme. This area continues to capture the excitement of the international mathematics community. Whenever interest momentarily wanes, there are new discoveries by the physicists that lead to new equations whose solution spaces are used to study these four-dimensional manifolds. Conversely, the new constructions, computations, and analysis provided by the mathematicians yield new insight into the structure of space-time and help structure the physics.

The above notion that there may indeed be parallel universes may well be explained by the theory of foliations on manifolds, i.e., a decomposition of an *n*-dimensional manifold as the disjoint union of submanifolds, called leaves, of lower dimension. For example, the three-dimensional sphere has a beautiful foliation by two-dimensional submanifolds: the Reeb foliation has as leaves copies of the Euclidean plane except for one leaf which is the torus. Foliations were well studied in the 1970s through the 1990s. However, their resurgence is imminent especially in the context of singular foliations, i.e., foliations of the complement of a prescribed singular set. For example, it is easy to show that amongst compact two-dimensional manifolds only the two-dimensional torus possesses a foliation with one-dimensional leaves. However, every two-manifold does have a foliation with one-dimensional leaves in the complement of a finite set of points. It is the behavior of these foliations at these singularities that gives rise to new insights into the geometry of surfaces and the study of three-dimensional manifolds. Likewise, the study of singular foliations may well structure the way in which we view our own universe, how we stack up with possible other parallel universes, and how gravity is that force that we can perceive yet floats freely between these universes. Better yet, these singular foliations may well provide new insight into the classification of four-dimensional manifolds.

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