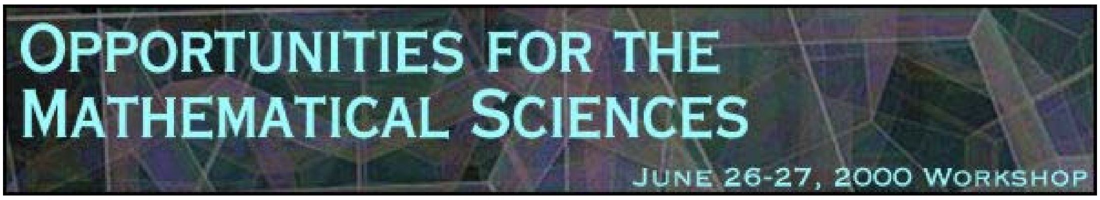

Random Matrix Theory, Quantum Physics, and Analytic Number Theory

A. GRANVILLE

Some of the most extraordinary cross-fertilization of ideas in recent mathematics comes in understanding the details of the distributions of primes, class numbers, ranks of elliptic curves, Frobenius eigenvalues in finite fields, etc., all important quantities that can be described in terms of the zeros of certain "zeta functions." Following up on earlier work of Montgomery and Dyson, Peter Sarnak of Princeton University has been pushing the idea that these spectra can be understood in terms of the distribution of eigenvalues from classes of random matrices drawn from certain classical groups, and with various collaborators, providing compelling evidence that this is so. Much of this is inspired by the work of Wigner, Mehta, and Dyson who used these same matrix models (which date back to Hermann Weyl) when modeling resonance lines of heavy nuclei (in quantum physics).

This new approach was inspired by Montgomery's work in the seventies, which determined the (Fourier transform of the) pair correlation function for pairs of zeros of the Riemann zeta-function in a limited range, and conjectured the pair correlation function in all ranges. At the time Dyson noted the analogies between Montgomery's results and conjectures, and a large body of work in quantum physics, but it was only in the late nineties that Rudnick and Sarnak were able to prove the generalization of Montgomery's results for n-level correlations. This led researchers to compute such correlation functions for many of the zeta functions of interest to number theory, and to determine the statistics of interest to physicists, and they quickly revealed much previously hidden structure. Although the computations are very suggestive, it is hard to concretely prove much. The one exception is the work of Katz and Sarnak on varieties over finite fields: starting from Deligne's great work on "equidistribution" of Frobenius eigenvalues for varieties over finite fields, they have proved that for many "families" the eigenvalues for the varieties are distributed just as the eigenvalues for certain classical groups. This is highly applicable work, which should reach into, for example, coding theory, and truly changes the limits of our understanding.

On the conjectural side, many young researchers have been using these ideas to delve into otherwise impenetrable questions, or to indicate how one might approach difficult problems. One exciting example is the very recent work of Conrey, Keating, Rubinstein and Snaith, who have shown that we would expect around $x^{3/4} (\log x)^{11/8}$ of the quadratic twists of a given elliptic curve to have rank bigger than one, a level of precision until recently unimaginable.

Last Modified: Oct 17, 2001

Previous page | Top of this page | Next page

The National Science Foundation, 4201 Wilson Boulevard, Arlington, Virginia 22230, USA
Tel: (703) 292-5111, FIRS: (800) 877-8339 | TDD: (800) 281-8749

Last Updated: 10/17/01
Text Only