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Model Theory and Tame Mathematics

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There are a number of ways in which modern logic affects mathematics, science and technology. This is maybe most obvious in the theory and practice of computation where the first rigorous models of computation were provided by the "recursion-theorists." One expects this to continue and deepen, especially at the level of software specification and verification.

Foundational studies throughout the twentieth century have given rise to many deep results showing the limits of mathematical reasoning. For example: there is no mechanical/effective way to enumerate all the truths of mathematics (Gödel), or even to decide whether an arbitrary system of polynomial equations over the integers has a solution in the integers (Matijasevich). Likewise the cardinality (size) of the set of real numbers can not be established within the accepted axioms of set theory (Gödel, Cohen). More generally, logic has pointed out many instances of wild, strange and pathological behavior in mathematics. There are still exciting developments in this genre, for example relating large cardinals to finite combinatorics and natural classes of functions.

However, I wish to discuss recent and possible future developments in model theory (a branch of mathematical logic) which have foundational imports of a rather different nature, in which general frameworks for understanding non-pathological behavior have been developed.

Abraham Robinson, who developed nonstandard analysis as well as the theory of model-completeness, was a pioneer of this kind of work. Other early work in this direction was Tarski's decision procedure for elementary Euclidean geometry [15]. This amounts to an effective procedure for deciding which (first order) statements about the structure (*R*, +,) are true. A by-product of his work was an identification of those subsets of R^n which are "first order definable" using just + and , as the sets defined by polynomial equations and inequalities. A key consequence is that definable subsets of the real line itself are finite unions of intervals and points. This feature subsequently (in the 1980's) became the definition of an o-minimal structure on *R* [4], [12]. Any o-minimal structure on *R* gives rise to a family of spaces (the definable sets), which is quite robust under various topological/geometric operations, and has local triviality, stratification, and uniform finiteness properties [5]. This seems to exactly fit Grothendieck's suggestion (in [6]) of finding an axiomatic development of "tame topology" or "tame spaces," in which the wildness of general topology is excluded, and the "topological properties of the various geometrical shapes" are at the fore [6]. Currently there is even an area of "o-minimal economics" [13]. We should say that applications to economics, via nonstandard analysis, were already obtained by Abraham Robinson.

In the early nineties it was proved that the exponential function lives in an o-minimal structure on *R* [16]. An elaboration of this result led to the solution of an old conjecture of Hardy regarding the asymptotic behavior of certain functions on *R* [5]. The search is on to find ever more richer o-minimal structures on the reals, and one hope is to solve Hilbert's sixteenth problem on limit cycles of polynomial planar vector fields in the process.

Let us informally define tame mathematics to be that part (of mathematics) which is *not* subject to the Gödel phenomena mentioned in the second paragraph. As we have seen above, elementary Euclidean geometry and more generally o-minimal geometry is part of tame mathematics. In fact, so are the auxiliary structures which were invented or given by nature, and surround number theory (such as the field of complex numbers, the p-adic fields, and the finite fields). The internal development of model theory over the past thirty years (stability theory and the classification of first order theories, [14] and later [11]) has given rise to a host of techniques and notions which could serve as a basis for organizing principles for tame mathematics, as well as yielding new results. Among these is a fundamental dichotomy between linear and nonlinear behavior. It has been conjectured and proved in a certain tame axiomatic context that "nonlinear" or even "nonsmooth" behavior of definable sets in a structure is explained by the presence of a definable field and thus the full richness of algebraic geometry [7]. "Linear" behavior is associated typically with finiteness results, such as the Faltings theorem - Mordell conjecture that a general polynomial in two variables with coefficients from *Q* has only finitely many rational zeros. The model-theoretic dichotomy result mentioned above yields analogous results in positive characteristic for which no other proof is known [9].

The linear/nonlinear dichotomy is valid in a number of contexts, such as ordinary differential and difference equations, or rather their respective algebraic frameworks, differential and difference fields, yielding strong consequences of "non-integrability" ([8], [1], [2]). Extensions to the infinite-dimensional context of partial differential equations are being worked on. One also hopes and expects the power of the stability-theoretic analyses to become available for all the structures surrounding number theory, and directly for the "tame" parts of number theory itself.

I have been discussing model theory, but there are other areas of logic, in particular descriptive set theory and the recent program of classifying "singular spaces" by their definable cardinalities, which give rise to novel interactions with other areas of mathematics [10].

In summary, the end of the twentieth century has witnessed qualitatively different connections between logic (or metamathematics) and mathematics, opening up many exciting opportunities for the twenty-first century.

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