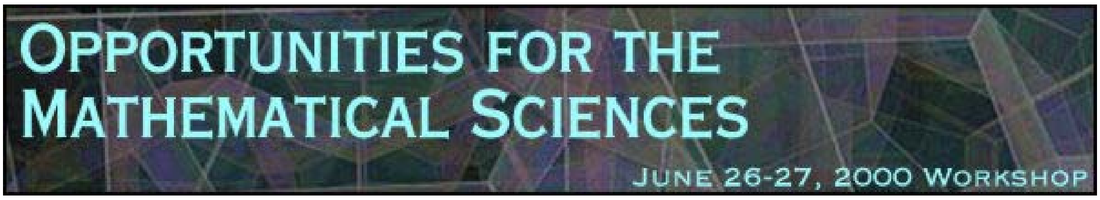

Research Opportunities in Nonlinear Partial Differential Equations

L. EVANS

I believe that a great development in mathematics for the new decades will be the continued rise of theory and applications for nonlinear partial differential equations. Unlike many highly evolved areas of classical mathematics, the general area of nonlinear PDE is mostly wide open. There are many fascinating and important nonlinear equations that we have only superficially studied. There are certainly significant, and perhaps not technically so difficult, discoveries to be made for most of these equations.

The last century saw the rise of linear analysis, motivated most strongly by quantum mechanics. But by now there is huge accumulated insight into nonlinear phenomena, largely justifying hopes that underlying are a few, basic governing mathematical concepts. Some of these have long been known (e.g. convexity inequalities, shock wave effects), and others are more recent (e.g. energy concentration, nonlinear averaging phenomena, etc.). The hope and expectation is that these mathematical principles can be better clarified, through a combination of rigorous theory, phenomena-based heuristics, and computer experimentation. The central issue for theoreticians in particular will be to move beyond perturbation theory, presumably to understand the proper notions of generalized solutions existing in the large.

SOME HIGHLIGHTS

1. Free boundary problems. Many PDE, primarily those describing phase transitions or fluid/air interfaces in physics and optimal stopping times in economics, entail unknown free boundaries. These are generally extremely difficult to handle rigorously, since the geometry of the unknown boundary affects the solution of the relevant PDE, and vice versa. The past twenty years have seen extraordinary progress, mostly towards showing that the free boundaries are "almost everywhere" smooth surfaces. This has been a deep application of geometric measure theory methods.

2. Dispersive equations and harmonic analysis. A major recent development has been the influx into nonlinear PDE theory of researchers trained in and inspired by classical harmonic analysis. The occasion has been the development of new Fourier transform methods, both to provide subtle, detailed estimates for the wave and Schrödinger operators and related linear PDE, and to handle certain natural polynomial nonlinearities.

3. Optimal transport. Another emerging area applies PDE methods for optimal "mass transport" problems. The basic issue is to find, among all mappings that rearrange a given measure into another, one which minimizes a cost functional. This seemingly very specific math problem turns out to have an extraordinary range of applications, to geometry, optimal design, stochastic models, and even fluid mechanics and meteorology. The highly nonlinear Monge-Ampere equation is at the heart of much of this.

4. Conservation laws. Nonlinear conservation laws record the basic physics for many systems, and these seemingly simple equations in fact support extraordinarily complicated solutions. A hallmark of these PDE is the advent of shock wave discontinuities, the structures of which are restricted by entropy inequalities of various types: we must accept discontinuous, weak solutions. This subject has always been very close to numerical analysis, both in taking computer simulations to suggest new theory and in providing rigorous convergence proofs for algorithms. With the recent breakthroughs on uniqueness of weak solutions, I expect substantial renewed progress.

5. Stochastic differential equations, continuum limits. Random ODE provide many interesting links to PDE theory, both linear and nonlinear. Among the most important are continuum limit problems for interacting stochastic particle systems, which have seen some major recent successes. Stochastic partial differential equations on the other hand have presented real challenges for theoreticians, largely owing to the singular structure of multidimensional "white noise".

6. Geometric motion. Many physical systems in appropriate asymptotic limits give rise to interfaces that evolve in time according to fairly simple geometric laws, for instance that the normal velocity equal the mean curvature. Such geometric evolution problems confront us with a spectrum of issues, from designing good numerical schemes to understanding changes in topological type. There is already a substantial literature on these topics, both theoretical and numerical, including applications to image processing, semiconductor etching, etc. Particularly useful has been the level set method of describing the moving interface in terms of an ambient "order parameter", which solves an appropriate nonlinear PDE.

7. Other geometric PDE. The past years have seen a great flowering of geometry, made possible at least in part by methods of nonlinear elliptic PDE, both single equations and systems. I expect this trend to continue, with perhaps more input from the theory of hyperbolic equations.

8. Dynamical methods in the calculus of variations. The PDE governing many nonequilibrium systems can be approximated by taking time to be discrete, and then solving a minimization problem over each time interval. There remain however profound problems in understanding the limit of the approximations as the time step goes to zero. I believe there is a great subject waiting here to be discovered, some sort of "time-dependent calculus of variations".

9. Kinetic formulations. The French PDE school has during the past decade pioneered a fascinating "kinetic" approach to nonlinear transport equations, based upon analogies with the classical passage from the Boltzmann equation to fluid mechanics. The physical procedure is still mathematically unjustified, but some related, and rigorous, procedures provide useful representation formulas for solutions of various nonlinear transport PDE, in terms of functions of more variables ("velocities").

10. Viscosity solutions. The notion of "viscosity solutions" has provided a robust and extremely flexible collection of tools for understanding weak solutions of certain highly nonlinear PDE that satisfy a maximum principle. The biggest successes have been in justifying dynamic programming procedures in control theory, but other applications have included large deviation estimates, interface motions, Hamiltonian dynamics, etc.

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The National Science Foundation, 4201 Wilson Boulevard, Arlington, Virginia 22230, USA
Tel: (703) 292-5111, FIRS: (800) 877-8339 | TDD: (800) 281-8749